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A new class of nonlinear generalizations of the Schrödinger equation

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Abstract. We propose a new, nonlinear representation-independent generalized Schrödinger equation, satisfying the homogeneity condition, which is valid for arbitrary representations and arbitrary operators. This is a generalization of the equation recently proposed by Doebner and Goldin for the wavefunction written in Cartesian coordinates. The new model leads to simple *exact* solutions describing, for instance, the relaxation of the two-level system and the harmonic oscillator, both in Fock and in coherent states bases.

Recently, Doebner and Goldin [1] proposed the following nonlinear generalization of the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = \hat{H}\psi(x, t) + iD\hbar G\{\psi(x, t)\} \tag{1}$$

where

$$\hat{H} = -(\hbar^2/2m)\nabla^2 + V(\boldsymbol{x}) \tag{2}$$

is the standard Hamiltonian of a particle with mass m moving in a scalar potential $V(\boldsymbol{x})$, D is constant positive diffusion coefficient, and

$$G\{\psi(x, t)\} = \nabla^2\psi(x, t) + \frac{|\nabla\psi(x, t)|^2}{|\psi(x, t)|^2}\psi(x, t) \tag{3}$$

is a nonlinear functional.

This is not the first and unique nonlinear model. Among others we have, for instance, the so-called 'standard nonlinear Schrödinger equation', where the leading term in the nonlinear functional $G\{\psi\}$ is proportional to $|\psi|^2\psi$ [2]. Various other nonlinear modifications of the Schrödinger equation have been investigated over the last few years. Some of them were constructed with the special goal of describing the friction phenomena in quantum mechanics [3–8], while other nonlinear equations were introduced objectively to test the fundamental postulates of quantum mechanics [9–18].

The outstanding feature of the Doebner–Goldin equation (1)–(3), is its group theoretical grounds: the nonlinear term was not simply added to the standard Schrödinger Hamiltonian in order to describe some restricted class of phenomena, but was derived

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from the analysis of possible representations of a quantum kinematical group. Moreover, slightly modified equations containing several different combinations of nonlinear terms like $(\nabla\psi/\psi)^2$ or $(\nabla\psi^*/\psi^*)^2$ [19] arise also in other fields, such as plasma physics and nonlinear optics [20]. Recently, we showed that for the forced harmonic oscillator placed in a uniform magnetic field, (1)–(3) describe, in a quite natural way, the relaxation of Gaussian wavepackets to the ground state [21–23].

The nonlinear functional (3) preserves the normalization of the wavefunction (due to the presence of the linear term $\nabla^2\psi$). Moreover, it belongs to the important class of functionals satisfying the 'homogeneity condition' [16]: if ψ is a solution to (1), then $\gamma\psi$ is also a solution for an arbitrary constant γ. The only drawback of functional (3) is its limited range of applications, since it is 'strongly tied' to the coordinate representation of the wavefunction. Thus, one of the best features of quantum mechanics—the possibility to choose any representation and any operators—is apparently lost in equations (1)–(3). In many problems the Cartesian coordinates and operators like ∇ are not the most convenient; for example, in quantum optics usually the bosonic creation and annihilation operators are preferred in order to deal with the Glauber coherent states [24].

Here we present a further generalization of the Doebner–Goldin equation, permitting one to remove the coordinate representation dependence, so becoming valid for arbitrary operators and representations. To obtain the modified equation we simply replace the gradient operator ∇ by an arbitrary (linear) operator $\hat{Q}$, and the 'coordinate' wavefunction $\psi(x)$ by the image of the state vector $|\psi\rangle$ in an arbitrary 'z-representation' $\langle z|\psi\rangle$:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\langle z|\psi\rangle=\langle z|\hat{H}|\psi\rangle+\mathrm{i}D\hbar\left\{\langle z|\psi\rangle\frac{\langle\psi|\hat{Q}^+|z\rangle\langle z|\hat{Q}|\psi\rangle}{\langle\psi|z\rangle\langle z|\psi\rangle}-\langle z|\hat{Q}^+\hat{Q}|\psi\rangle\right\} \tag{4}$$

($\hat{Q}^+$ means the hermitially conjugated operator). The set of states $|z\rangle$ is supposed complete, i.e.

$$\int|z\rangle\langle z|\,\mathrm{d}\mu(z)=\hat{1} \tag{5}$$

with some measure $\mathrm{d}\mu(z)$. Then the important property of the original Doebner–Goldin equation

$$\int\langle\psi|z\rangle\langle z|G\{\psi\}\rangle\,\mathrm{d}\mu(z)=0 \tag{6}$$

still holds, ensuring the preservation of normalization. Choosing $\hat{Q}=\nabla-(\mathrm{i}e/\hbar c)\boldsymbol{A}(\boldsymbol{x})$ in the x-representation, we immediately recover the equation for a quantum system in a magnetic field with vector potential $\boldsymbol{A}(\boldsymbol{x})$, already considered in [23].

The most important feature of the new form (4) is that it can be applied to quantum systems that require 'discrete' representations as well as to others that need 'continuous' ones. To see how it works in the case of 'discrete' systems, consider the simplest example of the two-level system with eigen-energies $\pm\omega$, so that $\hat{H}=\omega\sigma_3$, where σ_3 is the Pauli matrix, and $\hbar=1$. Choosing

$$\hat{Q}=\tfrac{1}{2}\sigma_+=\begin{Vmatrix}0 & 1\\ 0 & 0\end{Vmatrix}\qquad\hat{Q}^+\hat{Q}=\begin{Vmatrix}0 & 0\\ 0 & 1\end{Vmatrix}\ . \tag{7}$$

and the states $|z\rangle$ being the 'up' $|_{\uparrow}\rangle$ and 'down' $|^{\downarrow}\rangle$ states, we obtain the following equations for the corresponding amplitudes $\psi_{\uparrow}=\langle_{\uparrow}|\psi\rangle$ and $\psi_{\downarrow}=\langle^{\downarrow}|\psi\rangle$:

$$\dot{\psi}_{\uparrow}=-\mathrm{i}\omega\psi_{\uparrow}+D\frac{|\psi_{\downarrow}|^2}{\psi_{\uparrow}^*} \tag{8}$$

$$\dot{\psi}_{\downarrow}=\mathrm{i}\omega\psi_{\downarrow}-D\psi_{\downarrow}. \tag{9}$$

The second (linear!) equation has the solution

$$\psi_{\downarrow}(t)=\psi_{\downarrow}(0)\exp[(\mathrm{i}\omega-D)t] \tag{10}$$

and the squared amplitude (population) of the up level obeys the equation that follows from (8)

$$\frac{\mathrm{d}}{\mathrm{d}t}|\psi_{\uparrow}|^2=2D|\psi_{\downarrow}|^2 \tag{11}$$

with solution

$$|\psi_{\uparrow}(t)|^2=1-|\psi_{\downarrow}(0)|^2\exp(-2Dt). \tag{12}$$

We see that this simple model provides a very nice description of the relaxation to the up level. We re-emphasize that solutions (10) and (12) are *exact*.

The second example relates to the harmonic oscillator with the Hamiltonian $\hat{H}=\omega\hat{a}^{+}\hat{a}$, $[\hat{a},\hat{a}^{+}]=\hat{1}$. Choosing $\hat{Q}=\hat{a}$ and $|z\rangle\equiv|n\rangle$, we obtain the equations for the amplitudes $\psi_n=\langle n|\psi\rangle$ in the discrete Fock basis $|n\rangle$, $n=0,1,\ldots,$

$$\dot{\psi}_n=-\mathrm{i}n\omega\psi_n+D\left\{\frac{(n+1)|\psi_{n+1}|^2}{\psi_n^*}-n\psi_n\right\}. \tag{13}$$

One can easily check that the solutions to this equation can be written in the form

$$\psi_n(t)=[P_n(t)]^{1/2}\exp(-\mathrm{i}n\omega t) \tag{14}$$

with positive functions $P_n(t)$ (populations of the levels) satisfying the set of *linear* equations

$$\dot{P}_n=2D\{(n+1)P_{n+1}-nP_n\} \tag{15}$$

which *exactly coincide* with the equations for diagonal elements of the density matrix, that follow from the standard master equation [25, 26]

$$\mathrm{d}\hat{\rho}/\mathrm{d}t=-\mathrm{i}[\hat{H},\hat{\rho}]+D(2\hat{a}\hat{\rho}\hat{a}^{+}-\hat{a}^{+}\hat{a}\hat{\rho}-\hat{\rho}\hat{a}^{+}\hat{a}). \tag{16}$$

The difference is in the behaviour of the off-diagonal elements of the density matrix. For example, from (16) one obtains

$$\dot{\rho}_{mn}=D\{2[(m+1)(n+1)]^{1/2}\rho_{m+1,n+1}-(m+n)\rho_{mn}\}$$

whereas (13) leads to the following equations for the functions $\rho_{mn}=\psi_m\psi_n^*$:

$$\dot{\rho}_{mn}=D\left\{\frac{(m+1)\rho_{m+1,m+1}}{\rho_{mm}}+\frac{(n+1)\rho_{n+1,n+1}}{\rho_{nn}}-(m+n)\right\}\rho_{mn}$$

(both equations have been written from the interaction viewpoint, i.e. omitting the term $\mathrm{i}\omega(n-m)\rho_{mn}$). The exact solutions of (15) were found and investigated in detail

in [27, 28], by means of introducing the characteristic function $F(\lambda, t)=\Sigma\,\lambda^n P_n(t)$, satisfying (because of (15)) the first-order linear partial differential equation with respect to the auxiliary parameter λ and dimensionless time $\tau=2Dt$

$$\frac{\partial F}{\partial \tau}-(1-\lambda)\frac{\partial F}{\partial \lambda}=0.$$

For an arbitrary initial function $F_0(\lambda)$ the solution is

$$F(\lambda, \tau)=F_0(1+(\lambda-1)\,\mathrm{e}^{-\tau}). \tag{17}$$

If $\tau\to\infty$, the function $F(\lambda, \tau)$ tends to $F_0(1)\equiv 1$, implying relaxation to the ground state:

$$P_0(\infty)=1 \qquad P_n(\infty)=0 \qquad n\neq 0.$$

For instance, any initial coherent state with Poissonian distribution of the occupation numbers

$$P_n(0)=\frac{|\alpha|^{2n}}{n!}\,\mathrm{e}^{-|\alpha|^2} \qquad F_0(\lambda)=\exp[(\lambda-1)|\alpha|^2]$$

remains coherent because of (17):

$$F(\lambda, \tau)=\exp[(\lambda-1)\,\mathrm{e}^{-\tau}|\alpha|^2]$$

$$P_n(t)=\frac{|\alpha_t|^{2n}}{n!}\exp[-|\alpha_t|^2] \qquad |\alpha_t|^2=|\alpha|^2\,\mathrm{e}^{-\tau}. \tag{18}$$

Now consider the same problem in the 'continuous' basis, i.e. choose

$$|z\rangle\equiv|x\rangle \qquad \hat{Q}=\hat{a}=(2\omega\hbar m)^{-1/2}(\omega m\hat{x}+\mathrm{i}\hat{p}) \tag{19}$$

$$\hat{H}=\hbar\omega\hat{a}^{+}\hat{a}+f(t)\hat{a}^{+}+f^{*}(t)\hat{a}. \tag{20}$$

One can easily check that in this case equation (4) admits *exact* solutions in the form of Glauber's coherent states [24]

$$\langle x|\,\alpha_t\rangle=\left(\frac{\omega m}{\pi\hbar}\right)^{1/4}\exp\left\{-\frac{\omega m}{2\hbar}x^2+\left(\frac{2\omega m}{\hbar}\right)^{1/2}\alpha_t x-\tfrac{1}{2}\alpha_t^2-\tfrac{1}{2}|\alpha_t|^2+\mathrm{i}\phi(t)\right\} \tag{21}$$

provided the time-dependent parameter α_t obeys the *linear* equation

$$\dot{\alpha}_t+(\mathrm{i}\omega+D)\alpha_t+\mathrm{i}f(t)=0 \tag{22}$$

with solution

$$\alpha_t=\exp[-(\mathrm{i}\omega+D)t]\left\{\alpha_0-\mathrm{i}\int_0^t\exp[(\mathrm{i}\omega+D)\tau]f(\tau)\,\mathrm{d}\tau\right\} \tag{23}$$

and phase $\phi(t)$ given by

$$\phi(t) = -\int_0^t \mathrm{Re}[f^*(\tau)\alpha_\tau]\,\mathrm{d}t. \tag{24}$$

If $f=0$, then solution (21) gives the same distribution over energy levels as (18).

We see that (4) leads, in a quite straightforward way, to the *exact* solution of the problem of the harmonic oscillator coherent states relaxation. Moreover, the nonlinear correction to the usual Schrödinger equation does not destroy the classical character of the coherent states in this example. This same result was obtained in the framework of the 'microscopic' approach (i.e. when one takes into account explicitly the interaction of the oscillator concerned with a 'heat bath' consisting of a great number of other oscillators, and solves the corresponding *linear* Schrödinger or Heisenberg equations) [29]. But in the last case calculations were much more involved, and they required the use of the Wigner–Weisskopf approximation.

Further generalizations of (4) are possible. For example, one can take instead of the pair of hermitially conjugated operators $\hat{Q}$ and $\hat{Q}^+$, a pair of quite independent operators $\hat{Q}$ and $\hat{Y}$. Moreover, one can use the nonlinear functionals which are sums of terms like that in (4), but with different pairs $\hat{Q}_j$ and $\hat{Y}_j$. Also, some interesting generalizations to the case of mixed quantum states described in terms of statistical operators or density matrices exist, which will be considered in future publications.

Some comments concerning the possible critical remarks on equations with nonlinear terms like that in (3) or (4) are worth making. The term similar to (3), but with *imaginary* diffusion coefficient and without the linear part $\nabla^2\psi$ was rejected by Kibble [12] for two reasons. First, if $V(x)=0$ in (2), then such a term violates the invariance of (1) with respect to Galilean transformations. But if we are concerned with models describing *relaxation* (i.e. time irreversible processes), then we cannot demand Galilean invariance, which includes symmetry with respect to change of the sign of time. Moreover, in the model (4) with operators $\hat{Q}$ like (7) or (13), which are designated mainly to describe processes in quantum optics, Galilean invariance has no meaning at all.

The second remark is related to the presence of the wavefunction in the denominator of expression (4). Of course, it is uncomfortable to have to perform division by the ψ-function, because the question of singularities immediately arises. However, we can look at this problem from another point of view. The nonlinear models of quantum friction proposed earlier [3–5] did not yield to genuine relaxation, since all the eigenstates of the usual Hamiltonian $\hat{H}$ appeared also exact solutions of the corresponding nonlinear modifications. But the wavefunctions of all energy eigenstates, excepting ground state, have nodes. This means that these functions *cannot* be solutions of nonlinear equations (3) or (4). Therefore *every* excited eigenstate of the Hamiltonian $\hat{H}$ turns out to be *unstable* in the presence of new nonlinear terms, and we obtain genuine relaxation.

Finally, the exact solutions demonstrated above look so attractive, that equation (4) seems worthy of further investigations of its properties and other applications.

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